Mathematicians of Gaussian Elimination

Joseph F. Grcar

Gaussian elimination is universally known as "the" method for solving simultaneous linear equations. As Leonhard Euler remarked, it is the most natural way of proceeding ("der natürlichste Weg" [Euler, 1771, part 2, sec. 1, chap. 4, art. 45]). Because Gaussian elimination solves linear problems directly, it is an important technique in computational science and engineering, through which it makes continuing, albeit indirect, contributions to advancing knowledge and to human welfare. What is *natural* depends on the context, so the algorithm has changed many times with the problems to be solved and with computing technology.

Gaussian elimination illustrates a phenomenon not often explained in histories of mathematics. Mathematicians are usually portrayed as "discoverers", as though progress is a curtain that gradually rises to reveal a static edifice which has been there all along awaiting discovery. Rather, Gaussian elimination is living mathematics. It has mutated successfully for the last two hundred years to meet changing social needs.

Many people have contributed to Gaussian elimination, including Carl Friedrich Gauss. His method for calculating a special case was adopted by professional hand computers in the nineteenth century. Confusion about the history eventually made Gauss not only the namesake but also the originator of the subject. We may write *Gaussian elimination* to honor him without intending an attribution.

This article summarizes the evolution of Gaussian elimination through the middle of the twentieth century [Grcar, 2011a,b]. The sole development in ancient times was in China. An independent origin in modern Europe has had three phases. First came the schoolbook lesson, beginning with Isaac Newton. Next were methods for professional hand computers, which began with Gauss, who apparently was inspired by work of Joseph-Louis Lagrange. Last was the interpretation in matrix algebra by several authors, including John von Neumann. There may be no other subject that has been molded by so many renowned mathematicians.

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Bildarchiv Preussischer Kulturbresitz/Art Resource/NY. Photo by Olaf M. Tebmer.

Figure 1. It began in cuneiform tablets like VAT 8389. Vorderasiatisches Museum in Berlin, 12.1 by 7.9 cm.

Ancient Mathematics

Problems that can be interpreted as simultaneous linear equations are present, but not prevalent, in the earliest written mathematics. About a thousand cuneiform tablets have been published that record mathematics taught in the Tigris and Euphrates valley of Iran and Iraq in 2000 BC [Robson, 2008, table B.22]. Most tablets contain tables of various kinds, but some tablets are didactic (Figure 1). The first problem on VAT 8389 asks for the areas of two fields whose total area is 1800 sar, when the rent for one field is 2 silà of grain per 3 sar, the rent for the other is 1 silà per 2 sar, and the total rent on the first exceeds the other by 500 silà. If you do not remember these numbers, then you are not alone. The author of the tablet frequently reminds readers to "keep it in your head" in the literal translation by Høyrup [2002, pp. 77–82].[1] For perspective, when problems on these tablets can be written as simultaneous equations, usually one equation is nonlinear [Bashmakova and Smirnova, 2000, p. 3], which suggests that linear systems were not a large part of the Babylonian curriculum.

Simultaneous linear equations are prominent in just one ancient source. The *Jiuzhang Suanshu*, or *Nine Chapters of the Mathematical Art*, is a collection of problems that were composed in China over 2000 years ago [Shen et al., 1999] (Figure 2). The first of eighteen similar problems in chapter 8 asks for the grain yielded by sheaves of rice stalks from three grades of rice paddies. The combined yield is 39 dou of grain for 3 sheaves from top-grade paddies, 2 from medium-grade, and 1 from low-grade; and similarly for two more collections of sheaves. Mathematicians in ancient China made elaborate calculations using counting rods to represent numbers, which they placed inside squares arranged in a rectangle called a counting table (an ancient spreadsheet essentially). The right column in the following rectangle represents the first collection of paddies in the problem.

1	2	3	top
2	3	2	medium
3	1	1	low
26	34	39	yield

The solution was by Gaussian elimination. The right column was paired with each of the other columns to remove their top numbers, etc. The arithmetic retained integers by multiplying each column in a pair by the number atop the other and then subtracting right from left.

		3
4	5	2
8	1	1
39	24	39

Hart [2011, pp. 70–81] explains the entire calculation.

The *Nine Chapters* and similar, apparently derivative, work in Asia in later centuries [Hart, 2011; Libbrecht, 1973; Martzloff, 1997] are the only treatments of general linear problems until early modern Europe. The main surviving work of Greek and Roman algebra is the *Arithmetica* problem book by Diophantus, which is believed to be from the third century. Problem 19 in chapter (or book) 1 is to find four numbers such that the sum of any three exceeds the other by a given amount [Heath, 1910, p. 136]. Diophantus reduced the repetitive conditions to one with a new unknown, such as the

Figure 2. A depiction of Liu Hui, who said the *Nine Chapters* were already old when he wrote explanations of them in the third century.

sum of all the numbers, from which the other unknowns can be found. The earliest surviving work of ancient Hindu mathematics is the *Āryabhaṭīya* of Āryabhaṭa from the end of the fifth century. His linear problems are reminiscent of Diophantus but more general, being for any quantity of unknowns. Problem 29 in chapter 2 is to find several numbers given their total less each number [Plofker, 2009, p. 134]. The immediate sources for European algebra were Arabic texts. Al-Khwarizmi of Baghdad and his successors could solve the equivalent of quadratic equations, but they do not appear to have considered simultaneous linear equations beyond the special cases of Diophantus. For example, Rāshid [1984, p. 39] cites a system of linear equations that Woepcke [1853, p. 94, prob. 20] traces to a problem in the *Arithmetica*.

Schoolbook Elimination

Diophantus and Āryabhaṭa solved what can be interpreted as simultaneous linear equations without the generality of the *Nine Chapters*. An obvious prerequisite is technology able to express the problems of interest. The *Nine Chapters* had counting tables, and Europe had symbolic algebra. Possessing the expressive capability does not mean it is used immediately. Some time was needed to develop the concept of equations [Heeffer, 2011], and even then, of 107 algebras printed between 1550 and 1660 in the late Renaissance, only four books had *simultaneous* linear equations [Kloyda, 1938].

The earliest example found by Kloyda was from Jacques Peletier du Mans [1554]. He solved a problem of Girolamo Cardano to find the money held by three men when each man's amount plus a fraction of the others' is given. Peletier first took the approach of Cardano. This solution was by verbal reasoning in which the symbolic algebra is a

[1] *Exercise: Solve the problem in your head by any means and then consult Høyrup [2002, pp. 81–82] for the scribe's method. Discuss.*

convenient shorthand. The discourse has variables for just two amounts, and it represents the third by the given formula of the other two. Peletier [p. 111] then re-solved the problem almost as we do, starting from three variables and equations and using just symbolic manipulation (restated here with modern symbols):

$$\begin{aligned} 2R + A + B &= 64 \\ R + 3A + B &= 84 \\ R + A + 4B &= 124 \\ 2R + 4A + 5B &= 208 \\ 3A + 4B &= 144 \\ 3R + 4A + 2B &= 148 \\ 3R + 2A + 5B &= 188 \\ 6R + 6A + 7B &= 336 \\ 6R + 6A + 24B &= 744 \\ 17B &= 408 \end{aligned}$$

Peletier's overlong calculation suggests that removing unknowns systematically was a further advance. That step was soon made by Jean Borrel, who wrote in Latin as Johannes Buteo [1560, p. 190]. Borrel and the *Nine Chapters* both used the same double-multiply elimination process (restated with modern symbols):

$$\begin{aligned} 3A + B + C &= 42 \\ A + 4B + C &= 32 \\ A + B + 5C &= 40 \\ 11B + 2C &= 54 \\ 2B + 14C &= 78 \\ 150C &= 750 \end{aligned}$$

Lecturing on the algebra in Renaissance texts became the job of Isaac Newton (Figure 3) upon his promotion to the Lucasian professorship. In 1669–1670 Newton wrote a note saying that he intended to close a gap in the algebra textbooks: "This bee omitted by all that have writ introductions to this Art, yet I judge it very propper & necessary to make an introduction compleate" [Whiteside, 1968–1982, v. II, p. 400, n. 62]. Newton's contribution lay unnoticed for many years until his notes were published in Latin in 1707 and then in English in 1720. Newton stated the recursive strategy for solving simultaneous equations whereby one equation is used to remove a variable from the others.

> And you are to know, that by each Æquation one unknown Quantity may be taken away, and consequently, when there are as many Æquations and unknown Quantities, all at length may be reduc'd into one, in which there shall be only one Quantity unknown.
> — Newton [1720, pp. 60–61]

Newton meant to solve any simultaneous algebraic equations. He included rules to remove one variable from two equations which need not be linear: substitution (solve an equation for a variable and place the formula in the other) and equality-of-values (solve in both and set the formulas equal).

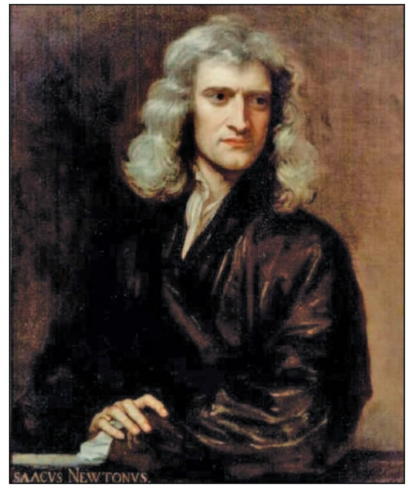


Figure 3. Isaac Newton (1643–1727) closed a gap in algebra textbooks. 1689 portrait by Godfrey Kneller.

While Newton's notes awaited publication, Michel Rolle [1690, pp. 42ff.] also explained how to solve simultaneous, specifically linear, equations. He arranged the work in two columns with strict patterns of substitutions. We may speculate that Rolle's emphasis survived in the "method of substitution" and that his "columne du retour" is remembered as "backward" substitution. Nevertheless, Newton influenced later authors more strongly than Rolle.

In the eighteenth century many textbooks appeared "all more closely resembling the algebra of Newton than those of earlier writers" [Macomber, 1923, p. 132]. Newton's direct influence is marked by his choice of words. He wrote "extermino" in his Latin notes [Whiteside, 1968–1982, v. II, p. 401, no. 63] that became "exterminate" in the English edition and the derivative texts. A prominent example is the algebra of Thomas Simpson [1755, pp. 63ff.]. He augmented Newton's lessons for "the Extermination of unknown quantities" with the rule of addition and/or subtraction (linear combination of equations).

Among many similar algebras, Sylvestre Lacroix (Figure 4) made an important contribution to the nomenclature. His polished textbooks presented the best material in a consistent style [Domingues, 2008], which included a piquant name for each concept. Accordingly, Lacroix [1804, p. 114] wrote, "This operation, by which one of the unknowns is removed, is called *elimination*" (Cette opération, par laquelle on chasse une des inconnues, se nomme *élimination*). The first algebra printed in the United States was a translation by John Farrar of Harvard College [Lacroix, 1818]. As derivative texts were written, "this is called elimination" became a fixture of American algebras.

Gaussian elimination for the purpose of schoolbooks was thus complete by the turn of the nineteenth century. It was truly *schoolbook* elimination,

J. G. Reinis, The Portrait Medallions of David D'Angers.

Figure 4. Sylvestre Lacroix (1765–1843) called it *élimination*. Image of an 1841 bas-relief by David d'Angers.

Figure 5. Carl Friedrich Gauss (1777–1855) devised the first professional method, replacing "common elimination". Lithograph by Siegfried Detlev Bendixen for the 1828 frontispiece of *Astronomische Nachrichten*.

because it had been developed to provide readily undertaken exercises in symbolic algebra.

Professional Elimination

A societal need to solve simultaneous linear equations finally arose when Adrien-Marie Legendre [1805] and Gauss [1809] (Figure 5) invented what Legendre named the method of least squares ("méthode des moindres quarrés," modern "carrés"). It was a method to draw statistical inferences for the unknowns in overdetermined, simultaneous linear equations by minimizing the sum of the squares of the discrepancies. Gauss became a celebrity by using unstated methods to calculate the orbit of the "lost" dwarf planet Ceres. Then Legendre succinctly stated what is now called the general linear model, which led to an unfortunate priority dispute once Gauss disclosed his calculations [Stigler, 1986]. The approximate solutions in the least-squares sense were obtained from Legendre's "equations of the minimum" or Gauss's "normal equations", which could be solved, they respectively said, by "ordinary methods" or "common elimination", meaning schoolbook elimination.[2] Such was the importance of the least-squares method that soon Gaussian elimination would evolve with the technology of professional hand computation.

[2] *The quoted phrases are: "d'équations du minimum" and "méthodes ordinares" [Legendre, 1805, p. 73], "eliminationem vulgarem" [Gauss, 1809, p. 214], and "Normalgleichungen" [Gauss, 1822, p. 84]. Following Gauss, the name normal equations is still given to the first-order differential conditions for the minimum.*

In modern notation of numerical analysis rather than statistics, for a matrix A of full column rank and suitably sized column vectors b and x, the least-squares problem is $\min_x \|b - Ax\|_2$, and the normal equations are $A^t Ax = A^t b$, to which elimination was applied.

Gauss himself was an incorrigible computer, who estimated that his prodigious calculations involved a million numbers [Dunnington, 2004, p. 138]. His least-squares publications mostly dealt with statistical justifications, but in one passage he described his own calculations. He reformulated the problem to reexpress the sum of squares in the canonical form of Lagrange [1759]. Gauss [1810, p. 22] wrote the overdetermined equations as

$$\begin{array}{l} n + a\,p + b\,q + c\,r + \ldots = w \\ n' + a'\,p + b'\,q + c'\,r + \ldots = w' \\ n'' + a''p + b''q + c''r + \ldots = w'' \\ \qquad\qquad \text{etc.,} \end{array}$$

where $p, q, r, \ldots$ are the unknown parameters of the linear model, while $n, a, b, c, \ldots$ are numbers that differ with each observed instance of the model (as indicated by the growing quantities of primes). The $w, w', w'', \ldots$ are the discrepancies whose sum of squares, Ω, is to be minimized. Gauss chose an unnamed notation,

$$[xy] = xy + x'y' + x''y'' + \ldots,$$

which he used to represent the numeric coefficients in the normal equations, equivalently, in the quadratic form Ω. Gauss then extended his bracket notation to

$$[xy,1] = [xy] - \frac{[ax][ay]}{[aa]},$$
$$[xy,2] = [xy,1] - \frac{[bx,1][by,1]}{[bb,1]},$$

and so on, in terms of which he constructed linear combinations of successively fewer parameters:

$$\begin{array}{lllll} A = [an] & + [aa]p & + [ab]q & + [ac]r & + \ldots \\ B = [bn,1] & & + [bb,1]q & + [bc,1]r & + \ldots \\ C = [cn,2] & & & + [cc,2]r & + \ldots \\ & & \text{etc.} & & \end{array}$$

These formulas complete the squares of successive parameters in the quadratic form, leaving

$$\Omega = \frac{A^2}{[aa]} + \frac{B^2}{[bb,1]} + \frac{C^2}{[cc,2]} + \cdots + [nn,\mu],$$

where μ is the quantity of variables. Thus $A = 0$, $B = 0$, $C = 0$, ... can be solved in reverse order to obtain values for $p, q, r, \ldots$, also in reverse order, at which Ω attains its minimum, $[nn,\mu]$.

Solving least-squares problems by Gauss's method required the numbers $[xy,k]$ for increasing indices k. Gauss halved the work of schoolbook elimination, because he needed x, y only in alphabetic order. More importantly, his notation discarded the equations of symbolic algebra, thereby freeing computers to organize the work more efficiently. When Gauss calculated, he simply wrote down lists of numbers using his bracket notation to identify the values [Gauss, 1810, p. 25].

The subsequent evolution of Gaussian elimination illustrates the meta-disciplinary nature of mathematics [Grcar, 2011c]. People such as Gauss with significant knowledge of fields besides mathematics were responsible for these developments. The advances do not superficially resemble either schoolbook elimination or what is taught at universities today, but they are no less important, because through them social needs were met.

Professional elimination began to develop due to the economic and military value of cartography. Gauss took government funds to map the German principality where he lived. Maps were drawn from networks of triangles using angles that were measured by surveyors. The angles had to be adjusted to make the triangles consistent. To that end, Gauss [1826] devised a method to find least-squares solutions of *under*determined equations by a quadratic-form calculation similar to the overdetermined case.[3] Once Friedrich Bessel [1838] popularized Gauss's approach, cartographic bureaus adopted the bracket notation. Gauss's calculations became part of the mathematics curriculum for geodesists.

> By whatever method of elimination is performed, we shall necessarily arrive at the same final values ...; but when the number of equations is considerable, the method of substitution, with Gauss's convenient notation, is universally followed.
>
> — Chauvenet [1868, p. 514] (Figure 6)

The first innovations after Gauss came from Myrick Doolittle [1881] (Figure 7). He was a computer at the United States Coast Survey who could solve forty-one normal equations in a week. This

Provided by the family of Adele Chauvenet Hanifin. Courtesy also of the United States Naval Academy.

Figure 6. William Chauvenet (1820–1870) and others taught Gauss's computing methods. Portrait circa 1839 by Eunice Makepeace Towle.

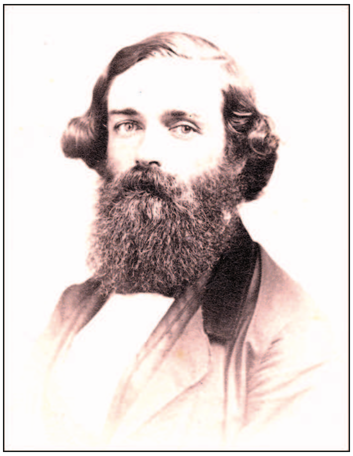
feat was astounding, because in principle it requires about $23000 \approx n^3/3$ arithmetic operations for $n = 41$. Doolittle dispensed with Gauss's brackets and instead identified the numbers by their positions in tables. He replaced the divisions in the bracket formulas by reciprocal multiplications. Doolittle's tables colocated values to help him use the product tables of August Leopold Crelle [1864]. An unrecognized but prescient feature of Doolittle's work combined graphs with algebra to reduce computational complexity. He used the cartographic triangulation to suggest an ordering of the overdetermined equations (equivalently, an arrangement of coefficients in the normal equations) to preserve zeroes in the work. The zeroes obviated many of the 23000 operations.

The next innovations were made by the French military geodesist André-Louis Cholesky [Benoit, 1924] (Figure 8). He, too, addressed the normal equations of the underdetermined, angle-adjustment problem. Although Cholesky is remembered for the square roots in his formulas,[4] his innovation was to order the arithmetic steps to exploit a feature of multiplying calculators. The machines were mass-produced starting in 1890 [Apokin, 2001], and Cholesky personally used a Dactyle (Figure 9). A side effect of long multiplication is that the machines could internally accumulate sums of products.[5] Calculations thus arranged were quicker to perform because fewer

[3] *For A of full row rank, these problems were $\min_{Ax=b} \|x\|_2$, where x are the angle adjustments and $Ax = b$ are the consistency conditions. The solution was given by correlate equations $x = A^t u$, where $AA^t u = b$ are again normal equations to be solved.*

[4] *Cholesky originated a construction that is now interpreted as decomposing a symmetric and positive definite matrix into a product LL^t, where L is a lower triangular matrix and ^{t} is transposition.*

[5] *The sums of products became known as dot-, inner-, or scalar-products of vectors after the adoption of matrix notation. Calculations employing sums were described as "abbreviated" or "compact".*



Courtesy of Antiochiana, Antioch College.

Figure 7. Myrick Hascall Doolittle (1830–1911), circa 1862.

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Figure 8. André-Louis Cholesky (1875–1918), circa 1895.

intermediate results had to be recorded on paper. Cholesky's tables for n equations held only $\mathcal{O}(n^2)$ numbers compared to Doolittle's $\mathcal{O}(n^3)$.

Statistical analysis in the form of regression analysis became the primary use for least-squares problems after the First World War. Doolittle's method was so prominent that it successfully made the transition to statistics. For example, United States government employees insisted on using Doolittle's method when they moved from the Coast Survey to the Department of Agriculture [Tolley and Ezekiel, 1927]. The department used least-squares methods for regression analyses in econometrics [Fox, 1989]. Cholesky's method was superior, but it received little notice [Brezinski, 2006] because it appeared in geodesy after statistics had begun to develop its own methods. The "square root method" did become widely used [Laderman, 1948] once it was reinvented by the American statistician Paul Dwyer [1945].

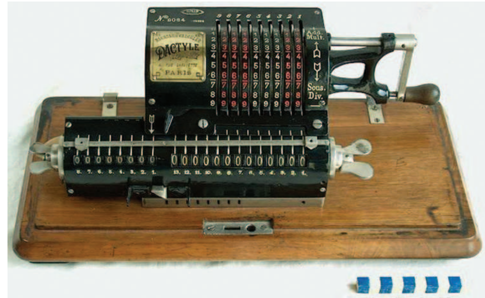


Photo by Freddy Haeghens, f.haeghens@skynet.be.

Figure 9. Dactyle calculator based on the pinwheel design of the Russian-Swedish entrepreneur Willgodt Odhner. Dactyle resold machines made by Brunsviga in Braunschweig and from 1905 by Chateau Bros. in Foncine-le-Haut.

The methods of Gauss, Doolittle, and Cholesky sufficed only for normal equations. In modern terms, the matrix of coefficients must be symmetric and positive definite. The need to routinely solve more kinds of simultaneous linear equations gradually developed in engineering. Unaware of the earlier work by Cholesky, a mathematician at the Massachusetts Institute of Technology, Prescott Crout [1941] (Figure 10), reorganized schoolbook Gaussian elimination to accumulate sums of products. Crout's method was publicized by a leading manufacturer of calculating machines (Figure 11).

Matrix Interpretation

The milieu of using symbolic algebra to modify Gaussian elimination ended with the adoption of matrix algebra. Several authors had developed matrices in the second half of the nineteenth century [Hawkins, 1975, 1977a,b, 2008]. Although matrices were not needed to compute by hand, the new representational technology showed that all the proliferating elimination algorithms were trivially related through matrix decompositions. Eventually matrices would help organize calculations for the purpose of programming electronic computers.

This development leads from the astronomical observatory of the Jagiellonian University to the numerical analysis textbooks that are presently in your campus bookstore. Manual computing motivated astronomer Tadeusz Banachiewicz [1938a,b] (Figure 12) to independently invent matrices in the form called Cracovians. They have a column-by-column product, which is the natural way to calculate with columns of figures by hand.

> It must, however, be conceded that in practice it is easier to multiply column by column than to multiply row by column …. It may, in fact, be said that the computations are made by Cracovians and the theory by matrices.
>
> — Jensen [1944]

Courtesy of MIT Museum.

Figure 10. Prescott Durand Crout (1907–1984), circa 1936.

Courtesy of Adam Strzałkowski.

Figure 12. Tadeusz Banachiewicz (1882–1954) used Cracovian algebra to describe computing methods. Photo from 1946.

Courtesy of John Wolff, www.vicnet.net.au/~wolff.

Figure 11. Crout's method was performed with machines such as this Marchant calculator, model 10ACT. It was manufactured in Oakland, California, in the 1930s and 1940s.

Banachiewicz advocated using Cracovians to represent calculations as early as 1933. The idea was realized in Arthur Cayley's matrix algebra by two people. Henry Jensen [1944] (Figure 13) of the Danish Geodætisk Institut used pictograms, □ = ◺◹, to emphasize that three algorithms for solving normal equations amounted to expressing a square matrix as a triangular product: the "Gaussian algorithm" (the calculation with Gauss's brackets), the Cracovian method, and Cholesky's method. A noteworthy aspect of Jensen's presentation was suggested by Frazer et al. [1938]: to represent arithmetic operations through multiplication by "elementary matrices". In the same year Paul Dwyer [1944] (Figure 14) of the University of Michigan showed that Doolittle's method was an "efficient way of building up" some "so-called triangular" matrices. He found no similar interpretation except in the work of Banachiewicz. The coincident papers of Jensen and Dwyer are the earliest to depict Gaussian elimination in roughly the modern form, that is, in terms of Cayleyan matrices.

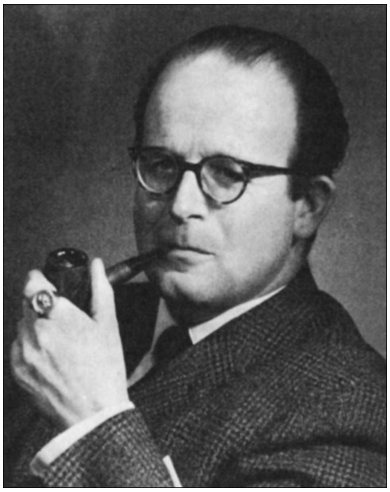
A deep use for the matrix interpretation came from John von Neumann (Figure 15) and his collaborator Herman Goldstine. They and others were in the process of building the first programmable, electronic computers. Concerns over the efficacy of the machines motivated von Neumann and Goldstine to study Gaussian elimination. The initial part of their analysis introduced the matrix decomposition.

> We may therefore interpret the elimination method as … the combination of two tricks: First, it decomposes A into a product of two [triangular] matrices … [and second] it forms their inverses by a simple, explicit, inductive process.
>
> — von Neumann and Goldstine [1947]

Von Neumann and Goldstine used matrix algebra to establish bounds on the rounding errors of what they anticipated would be the mechanized algorithm once computers became available. When the matrix is symmetric and positive definite, their bound remains the best that has been achieved. Although Gaussian elimination is observed to be accurate, a comparable error bound has yet to be established in the general case.[6]

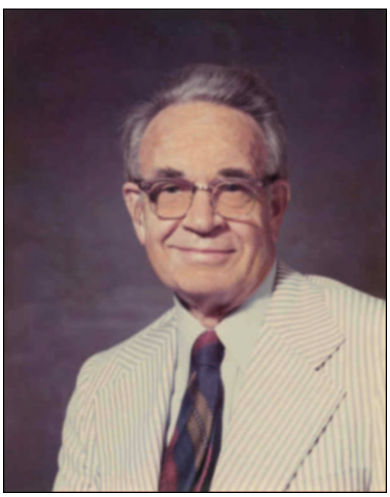
The next step to the campus bookstore was aided by *Mathematical Reviews*. John Todd (Figure 16) found Jensen's paper through *MR* and

[6] *No exercise: The end result of much study is, do not try to prove what von Neumann did not.*



Courtesy of the University of Copenhagen (Andersen 1975).

Figure 13. Henry Jensen (1915–1974) was inspired by Banachiewicz to use Cayleyan algebra.

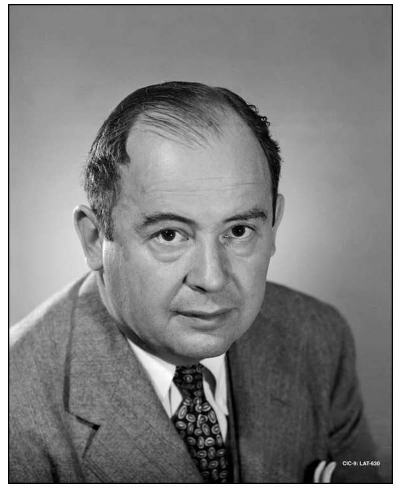


Courtesy of Prof. Ingram Olkin, Dept. of Statistics, Stanford University.

Figure 14. Paul Sumner Dwyer (1901–1982) independently used matrix algebra. Photo circa 1960s.

lectured on it at King's College London [Taussky and Todd, 2006]. An auditor communicated the matrix formalism to staff at the National Physical Laboratory. Among them was Alan Turing (Figure 17), who evidently learned of the matrix interpretation both from Jensen through Todd and also during a visit to von Neumann and Goldstine, whom Turing [1948] cited. He described Gaussian elimination in the manner of von Neumann and Goldstine by treating the general case of schoolbook elimination and in the manner of Jensen with elementary matrices. Turing wrote with a brevity of expression that made ideas clear without overworking them.

The invention of electronic computers created a discipline that was at first populated by those



Courtesy of the Los Alamos National Laboratory Archives.

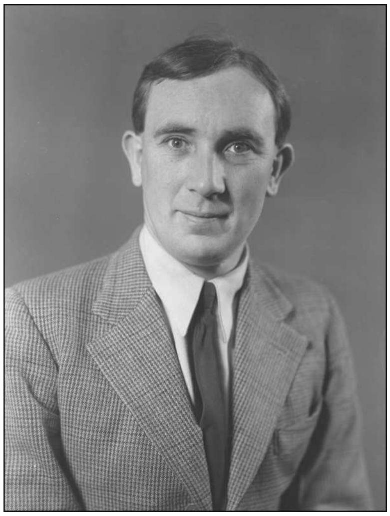
Figure 15. John von Neumann (1903–1957) saw "the combination of two tricks". Photo from March 1947.

who made scientific calculations [Traub, 1972; Wilkinson, 1970]. Among them, George Forsythe (Figure 18) was a visionary mathematician who is reputed to have named "computer science" [Knuth, 1972]. Gauss's involvement lent credence to the subject matter of the new discipline. The terminology that geodesists had used to describe the calculations of Gauss suggested an origin for what then was named simply "elimination". In an address to the American Mathematical Society, Forsythe [1953] misattributed "high school" elimination to Gauss and appears to have been the first to call it "Gaussian elimination" [Grcar, 2011a, tab. 1]. The name was widely used within a decade.

The university mathematics curriculum adopted matrix descriptions more slowly. Linear algebra itself was not commonly taught until the 1960s. When Fox [1964] and Forsythe and Moler [1967] wrote influential numerical analysis textbooks that featured the matrix interpretation, then they reprised Turing's presentation.

Coda

An *algorithm* is a series of steps for solving a mathematical problem. The matrix interpretation of Gaussian elimination seldom becomes an algorithm in a straightforward way, because the speed of computing depends on whether the calculation is well adapted to the problem and the computer. Just as Gauss developed the first professional method for least-squares calculations and then Doolittle developed a method for use with multiplication tables, other methods were developed more recently to solve the equations of finite-element analysis [Irons, 1970] with parallel computers [Duff and Reid, 1983]. While Cholesky and Crout emphasized sums of products for calculating machines, the arithmetic steps can be



Courtesy of the Archives, California Institute of Technology.

Figure 16. John Todd (1911–2007) lectured on Jensen's work in 1946.

Josè Mercado/Stanford News Service.

Figure 18. George Forsythe (1917–1972) called it "Gaussian elimination".

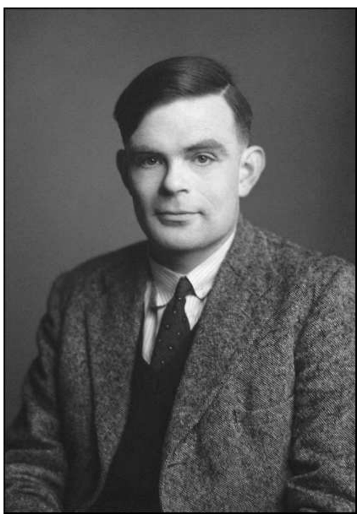


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Figure 17. Alan Turing (1912–1954) described it in the way universities now teach it. Photograph circa 1951.

reordered automatically to suit different computer architectures [Whaley and Dongarra, 1998]. More radical transformations are possible that reduce the work to solve n equations below $\mathcal{O}(n^3)$ arithmetic operations [Strassen, 1969; Cohn and Umans, 2003; Demmel et al., 2007]. Perhaps the only certainty about future algorithms is their name. Rather than being a Platonic archetype, Gaussian elimination is an evolving technique.

Acknowledgments

I am grateful to the editor and the referees for advice that improved this article. It is a pleasure to acknowledge the scholarship of Kloyda [1938], Macomber [1923], and Whiteside [1968–1982] (Figures 19, 20, 21), which made it possible to identify Newton's contribution.

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Figure 19. Sister Mary Thomas à Kempis Kloyda (1896–1977) discovered the few examples in Renaissance texts. She was a pioneering woman of American mathematics [Green and LaDuke, 2009]. Her dissertation is supported by an unrivaled collection of primary sources that is all the more impressive for being assembled before the age of electronic information.

Figure 20. Gertrude Louise Macomber [1923] observed that Newton's lesson for solving simultaneous equations was "the earliest appearance of this method on record". She anticipated the importance of historical antecedents to American textbooks and bested more prominent later academicians who neglected to examine the primary sources. Shasta County High School yearbook, Redding, California, 1920.

Figure 21. Derek Thomas Whiteside (1932–2008) [Guicciardini, 2008] remembered what Newton wrote in a monumental collection of his papers.

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